Introduction to symplectic topology: Gromov's nonsqueezing Theorem

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Abstract

This course is an introduction to symplectic topology via the surprising Gromov's nonsqueezing theorem. This theorem assert that if there is a symplectic embedding which maps the ball $B^{2n}(0,r)$ into the cylinder $B^2(0,R) \times \mathbb{R}^{2(n-1)}$ then $r \leq R$. The first course will be devoted to the introduction of some elementary properties of symplectic vector spaces and the proof of affine nonsqueezing theorem and the introduction of the linear symplectic width. The second course will be devoted to the introduction of symplectic manifolds and some of their immediate properties, namely, we will prove Darboux's theorem. In the last course, we will prove that the nonsqueezing theorem is equivalent to the existence of symplectic capacities and we will define the Hofer-Zehnder capacity and then prove Gromov's theorem. Finally, by using symplectic capacity we give a definition of symplectic homeomorphisms as a generalization of symplectic diffeomorphisms and hence give rise to symplectic topology.

1 Introduction

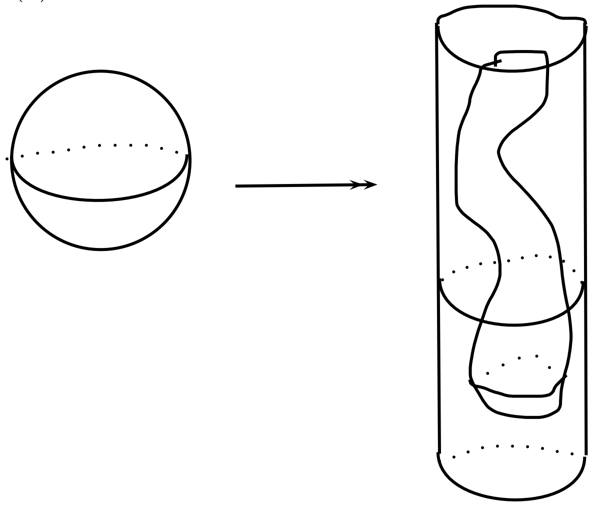
A **symplectic manifold** is a smooth manifold (M, ω) (eventually with a boundary) endowed with a closed nondegenerate differential 2-form ω. A smooth map $F : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ between two symplectic manifolds is called **symplectic** if $F^*\omega_2 = \omega_1$.
The standard model of a symplectic manifold is the Euclidean space $\mathbb{R}^{2n}$ endowed with its canonical symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the canonical linear coordinates of $\mathbb{R}^{2n}$. A **symplectomorphism** of $(\mathbb{R}^{2n}, \omega_0)$ is a diffeomorphism $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ such that $F^*\omega_0 = \omega_0$. It is obvious that a symplectomorphism F is also a preserving-volume diffeomorphism since $F^*\Omega = \Omega$ where $\Omega = \wedge^n \omega_0$ is a volume form associated to ω_0. Long time ago, many people used to believe that whatever could be done with a preserving-volume diffeomorphism could be done by a symplectomorphism (called **canonical transformation** by physicists). This belief was supported by the case $n = 1$ where both notions in fact coincide. When some people began to suspect that the group of symplectomorphisms is significantly smaller than the group of preserving-volume diffeomorphism there was no result to pinpoint the difference until Gromov proved his celebrated nonsqueezing theorem in 1985. This says that the standard closed symplectic ball cannot be symplectically embedded into a thin cylinder.
More precisely, the symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \leq R^2\right\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$ by $B^{2n}(r)$.



Theorem 1.1 *If there is a symplectic embedding $F : B^{2n}(r) \hookrightarrow Z^{2n}(R)$ then $r \leq R$.*

The Gromov's original proof used J-holomorphic curves [6]. In this course we give another proof of this theorem using the notion of symplectic capacity, namely, the symplectic capacity introduced by Hofer-Zehnder in [5]. Indeed, Gromov's nonsqueezing theorem gave rise to the following definition which is due to Ekeland and Hofer [2]. A **symplectic capacity** is a functor $\mathfrak{c}$ which assigns to every symplectic manifold (M, ω) a nonnegative (possibly infinite) number $\mathfrak{c}(M, \omega)$ and satisfies the following conditions.

- (**Monotonicity**) If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $\mathfrak{c}(M_1, \omega_1) \leq \mathfrak{c}(M_2, \omega_2)$.
- (**Conformality**) $\mathfrak{c}(M, \lambda\omega) = |\lambda| \mathfrak{c}(M, \omega)$.
- (**Non triviality**) $\mathfrak{c}(B^{2n}(1), \omega_0) > 0$ and $\mathfrak{c}(Z^{2n}(1), \omega_0) < \infty$.

The key to understanding symplectic capacities is the observation that the non triviality axiom makes it impossible for the volume of M to be a capacity. The requirement that $\mathfrak{c}(Z^{2n}(1), \omega_0)$ be finite means that these capacities are 2-dimensional invariants.
The existence of symplectic capacities is non trivial. In fact, we have the following proposition.

Proposition 1.1 *The existence of a symplectic capacity* $\mathfrak{c}$ *satisfying*

$$\mathfrak{c}(B^{2n}(1), \omega_0) = \mathfrak{c}(Z^{2n}(1), \omega_0) = \pi \tag{1}$$

is equivalent to Gromov's nonsqueezing theorem.

Proof. Suppose that there is a symplectic capacity satisfying (1) and suppose that there exists a symplectic embedding $(B^{2n}(r), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0)$. The monotonicity axiom implies

$$\mathfrak{c}(B^{2n}(r), \omega_0) \leq \mathfrak{c}(Z^{2n}(R), \omega_0).$$

Now it is easy to see that we have the following symplectic equivalences:

$$(B^{2n}(r), \omega_0) \simeq (B^{2n}(1), r^2\omega_0) \quad \text{and} \quad (Z^{2n}(R), \omega_0) \simeq (Z^{2n}(1), R^2\omega_0).$$

By the normality axiom we get that $r^2 \leq R^2$ and the Gromov's nonsqueezing theorem follows.
Conversely, suppose that the Gromov's nonsqueezing theorem holds. For any symplectic $2n$-dimensional manifold (M, ω), put

$$\mathfrak{c}_G(M, \omega) = \sup \mathrm{E}(M, \omega),$$

where

$$\mathrm{E}(M, \omega) = \left\{ \pi r^2 | \ (B^{2n}(r), \omega_0) \text{ embeds symplectically in } M \right\}.$$

Let us show that $\mathfrak{c}_G$ is a symplectic capacity satisfying (1). It is called **Gromov width**.
According to Darboux's theorem (see Theorem 3.1), there exists always a symplectic embedding of a closed symplectic ball $(B^{2n}(r), \omega_0)$ in (M, ω) and hence $\mathfrak{c}_G(M, \omega)$ is well-defined.
Suppose that there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$. Then

$$\mathrm{E}(M_1, \omega_1) \subset \mathrm{E}(M_2, \omega_2)$$

and hence $\mathfrak{c}_G(M_1, \omega_1) \leq \mathfrak{c}_G(M_2, \omega_2)$.
On the other hand, we have for any $\lambda \neq 0$,

$$\mathrm{E}(M, \lambda\omega) = \left\{ |\lambda|\pi r^2 | \; (B^{2n}(r), \omega_0) \text{ embeds symplectically in } (M, \omega) \right\}$$

and hence $\mathfrak{c}_G(M, \lambda\omega) = |\lambda| \mathfrak{c}_G(M, \omega)$.
Now it is obvious that $\mathfrak{c}_G(B^{2n}(r), \omega_0) = \pi r^2$. Moreover, the inclusion

$$(B^{2n}(R), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0)$$

is a symplectic embedding and hence $\mathfrak{c}_G(Z^{2n}(R), \omega_0) \geq \pi R^2$. On the other hand, if

$$(B^{2n}(r), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0)$$

is a symplectic embedding then according to Gromov's nonsqueezing theorem $r \leq R$ and hence $\mathfrak{c}_G(Z^{2n}(R), \omega_0) \leq \pi R^2$. Finally,

$$\mathfrak{c}_G(Z^{2n}(R), \omega_0) = \pi R^2,$$

and the proposition follows. $\square$

2 Affine nonsqueezing theorem

2.1 Symplectic vector spaces

Let $(e_1, \ldots, e_{2n})$ denote the canonical basis of $\mathbb{R}^{2n}$. The bilinear skew-symmetric 2-form

$$\omega_0 = \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^*$$

is non-degenerate, i.e.,

$$\omega_0(u, v) = 0 \quad \forall v \in \mathbb{R}^{2n} \implies u = 0.$$

The couple $(\mathbb{R}^{2n}, \omega_0)$ is the standard example of symplectic vector space. More generally, a **symplectic vector space** is a couple (V, ω) where V is finite dimensional $\mathbb{R}$-vector space and ω is a bilinear skew-symmetric 2-form on V which is nondegenerate. This means that ω satisfies:

1. ω is bilinear;
2. for any $u, v \in V$, $\omega(u, v) = -\omega(v, u)$;
3. for any $u \in V$,
$$\omega(u, v) = 0 \quad \forall v \in V \implies u = 0.$$

A symplectic vector space must be even dimensional. Indeed, if (V, ω) is a symplectic vector space and $(u_1, \ldots, u_n)$ is a basis of V, then the non-degeneracy of ω is equivalent to the fact that the matrix $(\omega(u_i, u_j))_{i,j=1}^n$ is invertible. Since a skew-symmetric odd dimensional real matrix must have vanishing determinant we deduce that n is even.

Let (V, ω) be a symplectic vector.

- A **linear symplectomorphism** of V is a vector space isomorphism $\Phi : V \longrightarrow V$ which preserves the symplectic form ω, i.e., for any $u, v \in V$,
$$\Phi^*\omega(u, v) := \omega(\Phi u, \Phi v) = \omega(u, v).$$
The linear symplectomorphisms of (V, ω) form a group which we denote by $\mathrm{Sp}(V, \omega)$. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$, we denote $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

- Let $W \subset V$ be a vector subspace. The **symplectic orthogonal** of W is the vector space
$$W^\omega = \{u \in V, \omega(u, v) = 0 \; \forall v \in W\}.$$

Proposition 2.1 *We have*
$$\dim W^\omega + \dim W = \dim V \quad \textit{and} \quad (W^\omega)^\omega = W.$$

Proof. We define $\imath : V \longrightarrow V^*$ by putting
$$\imath(v) = \omega(v, .),$$

where $\omega(v,.) : V \longrightarrow \mathbb{R}$, $u \mapsto \omega(v,u)$. The nondegeneracy of ω is equivalent to the fact that $\imath$ is bijective and we have $\imath(W^\omega) = W^0$ where W^0 is the annihilator of W, i.e.,

$$W^0 = \{\alpha \in V^*, \alpha(W) = 0\}.$$

Now, it is well-known that $\dim W^0 = \dim V - \dim W$ and the first formula follows.
It is obvious that $W \subset (W^\omega)^\omega$ and according to the first formula we have $\dim W = \dim(W^\omega)^\omega$ and hence $(W^\omega)^\omega = W$. $\Box$
A vector subspace W of V is called **isotropic** if $W \subset W^\omega$, **coisotropic** if $W^\omega \subset W$, **symplectic** if $W \cap W^\omega = \{0\}$, **Lagrangian** if $W = W^\omega$.

The following theorem is the main result of this subsection. It asserts that all symplectic vector spaces of the same dimension are symplectomorphic.

Theorem 2.1 *Let (V,ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $(e_1,\dots,e_n,\bar{e}_1,\dots,\bar{e}_n)$ such that*

$$\omega(e_i,e_j) = \omega(\bar{e}_i,\bar{e}_j) = 0 \quad \textit{and} \quad \omega(e_i,\bar{e}_j) = \delta_{ij}.$$

Such a basis is called a **symplectic basis**. *Moreover, there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \longrightarrow V$ such that*

$$\Phi^*\omega = \omega_0.$$

Proof. By induction over n. If $n = 1$, there exists obviously two vectors $e, \bar{e}$ such that $\omega(e,\bar{e}) = 1$ and hence $(e,\bar{e})$ is the desired basis.
Suppose that the result holds for n. Let (V,ω) be a $2n+2$-dimensional symplectic vector space. Since ω is nondegenerate there exists two vectors $e_1, \bar{e}_1$ such that $\omega(e_1,\bar{e}_1) = 1$. These vectors are linearly independent and span a 2-dimensional vector subspace W. Let us show that W is a symplectic vector subspace. Indeed, if $u \in W \cap W^\omega$ then $u = ae_1 + b\bar{e}_1$,

$$0 = \omega(u,e_1) = -b \quad \text{and} \quad 0 = \omega(u,\bar{e}_1) = a.$$

Hence $u = 0$. Thus, according to Proposition 2.1,

$$V = W \oplus W^\omega,$$

and moreover (W^ω,ω) is symplectic vector space. By induction hypothesis, there exists a symplectic basis of W^ω $(e_2,\dots,e_n,\bar{e}_2,\dots,\bar{e}_n)$. Finally, $(e_1,\dots,e_n,\bar{e}_1,\dots,\bar{e}_n)$ is symplectic basis of V.
The isomorphism $\Phi : \mathbb{R}^{2n} \longrightarrow V$ given by

$$\Phi(x_1,\dots,x_n,y_1,\dots,y_n) = \sum_{i=1}^n (x_i e_i + y_i \bar{e}_i)$$

satisfies $\Phi^*\omega = \omega_0$. $\square$

The **volume form** associated to a symplectic vector space (V, ω) is the $2n$-form given by

$$\Omega = \omega^n = \overbrace{\omega \wedge \ldots \wedge \omega}^{n} .$$

Note that $\Omega \neq 0$ and, more precisely, if $(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ is a symplectic basis then

$$\Omega = n!\,(e_1^* \wedge \bar{e}_1^* \wedge \ldots \wedge e_n^* \wedge \bar{e}_n^*) .$$

2.2 Linear symplectic group

In this subsection, we study the linear symplectomorphism group of a symplectic vector space in more detail. According to Theorem 2.1 it suffices to study the symplectomorphism group of $(\mathbb{R}^{2n}, \omega_0)$. Let $\mathbb{B}_0$ be the canonical basis of $\mathbb{R}^{2n}$ and $\langle\ ,\ \rangle$ the Euclidean inner product of $\mathbb{R}^{2n}$. The matrix of ω_0 in $\mathbb{B}_0$ is the matrix

$$\mathrm{J}_0 = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix} .$$

We have obviously $\mathrm{J}_0^2 = -\mathrm{I}_{2n}$,

$$\langle \mathrm{J}_0 u, \mathrm{J}_0 v \rangle = \langle u, v \rangle \quad \text{and} \quad \omega_0(u, v) = \langle \mathrm{J}_0 u, v \rangle. \tag{2}$$

It is easy to check that an isomorphism of $\mathbb{R}^{2n}$ is a linear symplectomorphism iff its matrix Φ in $\mathbb{B}_0$ satisfies

$$\Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0. \tag{3}$$

So we can identify $\mathrm{Sp}(2n)$ to the space of $2n \times 2n$-matrices which satisfy (3). If we write a $2n \times 2n$-matrix Φ as

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are real $n \times n$-matrices. It is straightforward to check that Φ satisfies (3) iff

$$A^{\mathrm{T}} C = C^{\mathrm{T}} A,\ B^{\mathrm{T}} D = D^{\mathrm{T}} B \quad \text{and} \quad A^{\mathrm{T}} D - C^{\mathrm{T}} B = \mathrm{I}_n. \tag{4}$$

Note first that a linear symplectomorphism preserves the volume form and hence its determinant is equal to 1. Thus

$$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n, \mathbb{R}) := \{\Phi \in \mathrm{GL}(2n, \mathbb{R}),\ \det \Phi = 1\} .$$

Note also that, according to (3), $\Phi \in \mathrm{Sp}(2n)$ iff $\Phi^{\mathrm{T}} \in \mathrm{Sp}(2n)$.
We identify $\mathrm{GL}(n,\mathbb{C})$ with a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ as follows

$$X + \imath Y \in \mathrm{GL}(n,\mathbb{C}) \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n,\mathbb{R}).$$

With this identification in mind, one can see easily that

$$\mathrm{GL}(n,\mathbb{C}) = \{\Phi \in \mathrm{GL}(2n,\mathbb{R}), \Phi \mathrm{J}_0 = \mathrm{J}_0 \Phi\}.$$

The unitary group is identified to

$$\mathrm{U}(n) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(n,\mathbb{C}), (X+\imath Y)(X-\imath Y)^{\mathrm{T}} = \mathrm{I}_n \right\}.$$

Lemma 2.1 *We have*

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{U}(n).$$

Proof. Let Φ be a $2n \times 2n$ real matrix. We have the following equivalence:

$$\begin{array}{rcl} \Phi \in \mathrm{GL}(n,\mathbb{C}) & \Longleftrightarrow & \Phi \mathrm{J}_0 = \mathrm{J}_0 \Phi, \\ \Phi \in \mathrm{Sp}(2n) & \Longleftrightarrow & \Phi^{\mathrm{T}} \mathrm{J}_0 \Phi = \mathrm{J}_0, \\ \Phi \in \mathrm{O}(2n) & \Longleftrightarrow & \Phi^{\mathrm{T}} \Phi = \mathrm{I}_{2n}. \end{array}$$

It is obvious that any of these conditions imply the third. Now, according to (4), the subgroup $\mathrm{Sp}(2n) \cap \mathrm{GL}(n,\mathbb{C})$ consists of this matrix

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n,\mathbb{R})$$

which satisfy

$$X^{\mathrm{T}} Y = Y^{\mathrm{T}} X \quad \text{and} \quad X^{\mathrm{T}} X + Y^{\mathrm{T}} Y = \mathrm{I}_n.$$

This is precisely the condition

$$(X+\imath Y)(X-\imath Y)^{\mathrm{T}} = \mathrm{I}_n.$$

□

For any $\Phi \in \mathrm{GL}(2n,\mathbb{R})$ we denote by $\sigma(\Phi) \subset \mathbb{C}$ the set of zeros of the characteristic polynomial associated to Φ, i.e.,

$$\lambda \in \sigma(\Phi) \quad \Longleftrightarrow \quad \det(\Phi - \lambda \mathrm{I}_{2n}) = 0.$$

For any $\lambda \in \sigma(\Phi)$ we denote by $m(\lambda)$ the multiplicity of λ as a zero of the characteristic polynomial. Note that

$$\sigma(\Phi) = \sigma(\Phi^{\mathrm{T}}) \quad \text{and} \quad \sigma(\Phi^{-1}) = \left\{\lambda^{-1}, \lambda \in \sigma(\Phi)\right\}.$$

Lemma 2.2 *Let $\Phi \in \mathrm{Sp}(2n)$. Then:*

1. $\sigma(\Phi) = \sigma(\Phi^{-1})$.

2. *If $\pm 1 \in \sigma(\Phi)$ then it occurs with even multiplicity.*

Moreover,

$$\Phi z = \alpha z, \quad \Phi z' = \lambda' z, \ \lambda\lambda' \neq 1 \quad \Longrightarrow \quad \omega_0(z, z').$$

Proof. Since $\mathrm{J}_0^2 = -\mathrm{I}_{2n}$, we get from (3)

$$\Phi^{\mathrm{T}} = \mathrm{J}_0 \Phi^{-1} \mathrm{J}_0^{-1}$$

and hence

$$\sigma(\Phi^{-1}) = \sigma(\Phi^{\mathrm{T}}) = \sigma(\Phi).$$

From this relation, we deduce that

$$\sum_{\lambda \in \sigma(\Phi), \lambda \neq \pm 1} m(\lambda) = 2p$$

with $p \in \mathbb{N}$. On the other hand, since

$$1 = \det \Phi = \prod_{\lambda \in \sigma(\Phi)} \lambda^{m(\alpha)}$$

we deduce that if $-1 \in \sigma(\Phi)$ then $m(-1)$ is even. Moreover, since the degree of the characteristic polynomial is even the eigenvalue 1 occurs with even multiplicity as well.
The last statement follows from the identity

$$\omega_0(\Phi z, \Phi z') = \omega_0(z, z') = \lambda\lambda' \omega_0(z, z').$$

$\square$

Let $P \in \mathrm{GL}(2n, \mathbb{R})$ symmetric and positive definite. Then P is diagonalizable in an orthonormal basis, i.e., there exists a matrix $Q \in \mathrm{O}(2n)$ such that

$$P = Q \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_{2n} \end{pmatrix} Q^{\mathrm{T}},$$

where $0 < \lambda_1 \leq \ldots \leq \lambda_{2n}$. For any real number $\alpha > 0$ put

$$P^\alpha = Q \begin{pmatrix} \lambda_1^\alpha & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_{2n}^\alpha \end{pmatrix} Q^{\mathrm{T}}.$$

It is an easy exercise to show that P^α does not depends on Q.

Lemma 2.3 *If $P = P^{\mathrm{T}} \in \mathrm{Sp}(2n)$ is symmetric, positive definite symplectic matrix then $P^\alpha \in \mathrm{Sp}(2n)$ for any real number $\alpha > 0$.*

Proof. We will show that, for any $z, z' \in \mathbb{R}^{2n}$,

$$\omega_0(P^\alpha z, P^\alpha z') = \omega(z, z'). \tag{$*$}$$

First, denote by $0 < \lambda_1 < \ldots < \lambda_r$ the different eigenvalues of P and $V_{\lambda_1}, \ldots, V_{\lambda_r}$ the corresponding eigenspaces. We have

$$\mathbb{R}^{2n} = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}.$$

We distinguish two cases:

- $z \in V_{\lambda_i}$, $z' \in V_{\lambda_j}$ and $\lambda_i \lambda_j \neq 1$. Then $P^\alpha z = \lambda_i^\alpha z$ and $P^\alpha z' = \lambda_j^\alpha z'$ and according to Lemma 2.2 $\omega_0(z, z') = 0$ and $(*)$ holds.

- $z \in V_{\lambda_i}$, $z' \in V_{\lambda_j}$ and $\lambda_i \lambda_j = 1$. Then $P^\alpha z = \lambda_i^\alpha z$, $P^\alpha z' = \lambda_j^\alpha z'$ and $(*)$ holds. □

Let us recall the polar decomposition of the linear group $\mathrm{GL}(n, \mathbb{R})$.

Theorem 2.2 *Let $A \in \mathrm{GL}(n, \mathbb{R})$. Then there exists an unique couple (O, S) such that*

$$A = SO,$$

where $O \in \mathrm{O}(n)$ and S is symmetric, positive definite.

Proposition 2.2 *The unitary group $\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$ and the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible.*

Proof. First let us prove that the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible. Now, according to Theorem 2.2, every matrix $\Phi \in \mathrm{Sp}(2n)$ can be uniquely decomposed as

$$\Phi = SO$$

where S is symmetric and positive definite and O is orthogonal. By the preceding lemma

$$S = (\Phi\Phi^{\mathrm{T}})^{\frac{1}{2}} \in \mathrm{Sp}(2n)$$

and hence

$$O = S^{-1}\Phi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \overset{\text{Lemma 2.1}}{=} \mathrm{U}(n).$$

Thus the map

$$\mathrm{Sp}(2n) \times [0,1] \longrightarrow \mathrm{Sp}(2n) \ : \ (\Phi, t) \mapsto (\Phi\Phi^{\mathrm{T}})^{-\frac{t}{2}}\Phi$$

is a retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$.

To see that $\mathrm{U}(n)$ is a maximal compact subgroup, let $\mathrm{G} \subset \mathrm{Sp}(2n)$ be any compact subgroup. We must show that G is conjugate to a subgroup of $\mathrm{U}(n)$. Wed define $P \in \mathrm{Sp}(2n)$ by

$$P = \int_{\mathrm{G}} g^{\mathrm{T}} g dg$$

where dg is the Haar measure of G. It is obvious that P is symmetric and positive definite. Moreover, we have, for any $\Phi \in \mathrm{G}$,

$$\begin{aligned}
\Phi^{\mathrm{T}} P \Phi &= \Phi^{\mathrm{T}} \left(\int_{\mathrm{G}} g^{\mathrm{T}} g dg \right) \Phi \\
&= \int_{\mathrm{G}} (g\Phi)^{\mathrm{T}} (g\phi) dg \\
&= P.
\end{aligned}$$

Since $P^{\frac{1}{2}}$ is a symplectic matrix we obtain

$$\Phi \in \mathrm{G} \quad \Longrightarrow \quad P^{\frac{1}{2}} \Phi P^{-\frac{1}{2}} \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \overset{\text{Lemma 2.1}}{=} \mathrm{U}(n).$$

This proves the proposition. □

2.3 The affine nonsqueezing theorem

An **affine symplectomorphism** of $\mathbb{R}^{2n}$ is a map $\phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of the formula

$$\phi(z) = \Phi z + z_0,$$

where $\Phi \in \mathrm{Sp}(2n)$ and $z_0 \in \mathbb{R}^{2n}$. We denote by $\mathrm{ASp}(2n)$ the group of affine symplectomorphisms. The affine nonsqueezing theorem asserts that a ball in $\mathbb{R}^{2n}$ can only be embedded into a symplectic cylinder by an affine symplectomorphism if it has a smaller radius. The symplectic cylinder of radius $R > 0$ is

$$Z^{2n}(R) = \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n},\ x_1^2 + y_1^2 \leq R^2\right\} \simeq B^2(R) \times \mathbb{R}^{2n-2}.$$

We denote the Euclidean closed ball of center 0 and the radius r in $\mathbb{R}^{2n}$ by $B^{2n}(r)$.

Theorem 2.3 *Let $\phi \in \mathrm{ASp}(2n)$ and assume that $\phi(B^{2n}(r)) \subset Z^{2n}(R)$. Then $r \leq R$.*

Proof. Write $\phi(z) = \Phi(z) + z_0$ with $\Phi \in \mathrm{Sp}(2n)$ and $z_0 \in \mathbb{R}^{2n}$ and denote by $(e_1, \ldots, e_{2n})$ the canonical basis of $\mathbb{R}^{2n}$. The condition $\phi(B^{2n}(r)) \subset Z^{2n}(R)$ is equivalent to

$$\forall u \in B^{2n}(r), \quad ((\Phi(u))_1 + z_0^1)^2 + ((\Phi(u))_{n+1} + z_0^{n+1})^2 \leq R^2. \qquad (*)$$

Now it is easy to see that

$$(\Phi(u))_1 = \langle \Phi^{\mathrm{T}} e_1, u \rangle \quad \text{and} \quad (\Phi(u))_{n+1} = \langle \Phi^{\mathrm{T}} e_{n+1}, u \rangle.$$

The crucial point is that since $\Phi^{\mathrm{T}} \in \mathrm{Sp}(2n)$,

$$\omega_0(\Phi^{\mathrm{T}} e_1, \Phi^{\mathrm{T}} e_{n+1}) = \omega_0(e_1, e_{n+1}) = 1.$$

So, by using (2) and the Cauchy-Schwarz inequality, we get

$$1 = \omega_0(\Phi^{\mathrm{T}} e_1, \Phi^{\mathrm{T}} e_{n+1}) \leq |\Phi^{\mathrm{T}} e_1||\Phi^{\mathrm{T}} e_{n+1}|.$$

This inequality implies that either $|\Phi^{\mathrm{T}} e_1|$ or $|\Phi^{\mathrm{T}} e_{n+1}|$ is greater than or equal to one. Assume without loss of generality that $|\Phi^{\mathrm{T}} e_1| \geq 1$ and choose in $(*)$ $u = \epsilon r \frac{\Phi^{\mathrm{T}} e_1}{|\Phi^{\mathrm{T}} e_1|}$ where ϵ is the sign of z_0^1. We get

$$r^2 \leq (r|\Phi^{\mathrm{T}} e_1| + |z_0^1|)^2 + ((\Phi(u))_{n+1} + z_0^{n+1})^2 \leq R^2,$$

and the theorem follows. □

The nonsqueezing property characterizes in fact linear symplectomorphisms. We call a subset $A \subset \mathbb{R}^{2n}$ a **linear symplectic ball** of **radius** r if there exists $\Phi \in \mathrm{Sp}(2n)$ such that $A = \Phi(B^{2n}(r))$. It results that A and $B^{2n}(r)$ must have the same volume and hence r does not depend on Φ. In a similar way, a subset $Z \in \mathbb{R}^{2n}$ is called **linear symplectic cylinder** if there exists $\Phi \in \mathrm{Sp}(2n)$ and $r > 0$ such that $Z = \Phi(Z^{2n}(r))$. It follows from

Theorem 2.3 that for any linear symplectic cylinder Z the number $r > 0$ is a linear symplectic invariant. Indeed, suppose that

$$Z = \Phi_1(Z^{2n}(r_1)) = \Phi_2(Z^{2n}(r_2))$$

with $\Phi_1, \Phi_2 \in \mathrm{Sp}(2n)$. Since $B^{2n}(r_1) \subset Z^{2n}(r_1)$ we deduce that

$$\Phi_2^{-1}\Phi_1(B^{2n}(r_1)) \subset Z^{2n}(r_2)$$

and by Theorem 2.3 $r_1 \leq r_2$. A similar argument gives $r_2 \leq r_1$ and hence $r_1 = r_2$.

A nonsingular $2n \times 2n$ matrix Φ is said to have the **linear nonsqueezing property** if for every linear symplectic ball B of radius r and every linear symplectic cylinder Z of radius R we have

$$\Phi(B) \subset Z \quad \Longrightarrow \quad r \leq R.$$

The following theorem shows that linear symplectomorphisms are characterized by the linear nonsqueezing property. More precisely, we must also include the case of **anti-symplectic** matrices Φ which satisfy $\Phi^*\omega_0 = -\omega_0$.

Theorem 2.4 *Let Φ be a non singular $2n \times 2n$ matrix such that Φ and Φ^{-1} have the linear nonsqueezing property. Then Φ is either symplectic or anti-symplectic.*

Proof. Assume that Φ is neither symplectic nor anti-symplectic. Then neither is Φ^{T} and so, by a density argument, there exists vector $u, v \in \mathbb{R}^{2n}$ such that

$$\omega_0(\Phi^{\mathrm{T}}u, \Phi^{\mathrm{T}}v) \neq \pm\omega_0(u, v).$$

Perturbing u and v slightly, and using the fact that Φ is nonsingular, we way assume that $\omega_0(u, v) \neq 0$ and $\omega_0(\Phi^{\mathrm{T}}u, \Phi^{\mathrm{T}}v) \neq 0$. Moreover, replacing Φ by Φ^{-1} if necessary, we may assume that $\omega_0(\Phi^{\mathrm{T}}u, \Phi^{\mathrm{T}}v) < \omega_0(u, v)$. Now, by rescaling u if necessary, we obtain

$$0 < \lambda^2 = \omega_0(\Phi^{\mathrm{T}}u, \Phi^{\mathrm{T}}v) < \omega_0(u, v) = 1.$$

Hence there exist symplectic bases $(u_1, v_1, \ldots, u_n, v_n)$ and $(u_1', v_1', \ldots, u_n', v_n')$ of $\mathbb{R}^{2n}$ such that

$$u_1 = u, \quad v_1 = v, \quad u_1' = \lambda^{-1}\Phi^{\mathrm{T}}u, \quad v_1' = \pm\lambda^{-1}\Phi^{\mathrm{T}}v.$$

Denote by $\Psi \in \mathrm{Sp}(2n)$ (resp. $\Psi' \in \mathrm{Sp}(2n)$) the matrix which maps the canonical basis of $\mathbb{R}^{2n}$ to $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ (resp. $(u_1', \ldots, u_n', v_1', \ldots, v_n')$). Then the matrix

$$A = \Psi'^{-1}\Phi^{\mathrm{T}}\Psi$$

satisfies

$$Ae_1 = \lambda e_1 \quad \text{and} \quad Af_1 = \pm\lambda f_1.$$

This implies that the transposed matrix A^{T} maps the unit ball $B^{2n}(1)$ to cylinder $Z^{2n}(\lambda)$. But since $\lambda < 1$ this means that Φ does not have the nonsqueezing property in contradiction to our assumption. This proves the theorem. $\square$

The affine nonsqueezing theorem gives rise to the notion of the **linear symplectic width** of an arbitrary subset $A \subset \mathbb{R}^{2n}$, defined by

$$\mathfrak{W}_L(A) = \sup\left\{\pi r^2 |\ \phi(B^{2n}(r)) \subset A \text{ for some } \phi \in \mathrm{ASp}(\mathbb{R}^{2n})\right\}.$$

It follows from Theorem 2.3 that the linear symplectic width has the following properties:

- (**Monotonicity**) If $\phi(A) \subset B$ for some $\phi \in \mathrm{ASp}(\mathbb{R}^{2n})$ then $\mathfrak{W}_L(A) \leq \mathfrak{W}_L(B)$.
- (**Conformality**) $\mathfrak{W}_L(\lambda A) = \lambda^2 \mathfrak{W}_L(A)$.
- (**Nontriviality**) $\mathfrak{W}_L(B^{2n}(r)) = \mathfrak{W}_L(Z^{2n}(r)) = \pi r^2$.

The nontriviality axiom implies that $\mathfrak{W}_L$ is a two-dimensional invariant. It is obvious from the monotonicity property that affine symplectomorphisms preserve the linear symplectic width. We shall prove that this property in fact characterizes symplectic and anti-symplectic linear maps.
Recall that an ellipsoid centered at 0 in the Euclidean space $\mathbb{R}^{2n}$ is given by

$$E = \left\{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} |\ \sum_{i,j=1}^{2n} a_{ij} x_i x_j \leq 1\right\}$$

where the $2n \times 2n$ matrix (a_{ij}) is symmetric positive definite. Define the inner product

$$\langle u, v\rangle_A = \langle Au, v\rangle,$$

where $\langle\ ,\ \rangle$ is the canonical inner product on $\mathbb{R}^{2n}$. Hence

$$u \in E \quad \Longleftrightarrow \quad \langle u, u\rangle_A \leq 1.$$

Since A is symmetric positive definite there exists an orthonormal basis $(u_1, \ldots, u_{2n})$ and a family or real numbers $0 < \lambda_1 \leq \ldots \leq \lambda_{2n}$ such that $Au_i = \lambda_i u_i$ for $i = 1, \ldots, 2n$. So, if Φ is the element of $\mathrm{O}(2n)$ which maps the canonical basis of $\mathbb{R}^{2n}$ to $(u_1, \ldots, u_{2n})$, we get

$$\Phi^{-1}(E) = \left\{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} |\ \sum_{i=1}^{2n} \frac{x_i^2}{\rho_i^2} \leq 1\right\},$$

where $\rho_i = \sqrt{\lambda_i^{-1}}$.

Symplectically an ellipsoid can be characterized as follows.

Lemma 2.4 *Given any ellipsoid*

$$E = \left\{ (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} x_i x_j \leq 1 \right\}$$

there is a linear symplectomorphism $\Phi \in \mathrm{Sp}(2n)$ *such that*

$$\Phi(E) = E(r) := \left\{ (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} | \sum_{j=1}^{n} \frac{x_j^2 + y_j^2}{r_j^2} \leq 1 \right\},$$

for some n-uple $r = (r_1, \ldots, r_n)$ *with* $0 < r_1 \leq \ldots \leq r_n$. *Moreover,* r *is entirely determined by* E.

Proof. Since ω_0 is nondegenerate there exists a skew-symmetric (with respect to $\langle\ ,\ \rangle_A$) nonsingular endomorphism J such that

$$\omega_0(u, v) = \langle Ju, v \rangle_A.$$

According to a classical result in linear algebra there exists an orthonormal basis of $\langle\ ,\ \rangle_A$ say $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ and a family of real number $0 < a_1 \leq \ldots \leq a_n$ such that, for $i = 1, \ldots, n$,

$$Ju_i = a_i v_i \quad \text{and} \quad Jv_i = -a_i u_i.$$

For $i = 1, \ldots, n$, put $u_i' = \sqrt{a_i^{-1}} u_i$ and $v_i' = \sqrt{a_i^{-1}} v_i$. It is easy to check that $(u_1', \ldots, u_n', v_1', \ldots, v_n')$ is a symplectic basis of $\mathbb{R}^{2n}$. Denote by Φ the element of $\mathrm{Sp}(2n)$ which maps the canonical basis to this basis. Now, we have

$$\begin{aligned}
\langle u, u \rangle_A &= \omega_0(J^{-1}u, u) \\
&= \sum_{i=1}^{n} \big(\omega_0(J^{-1}u, v_i')\omega_0(u_i', u) - \omega_0(J^{-1}u, u_i')\omega_0(v_i', u) \big) \\
&= \sum_{i=1}^{n} \big(\omega_0(J^{-1}v_i', u)\omega_0(\Phi e_i, u) - \omega_0(J^{-1}u_i', u)\omega_0(\Phi e_{n+1}, u) \big) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i} (\omega_0(u_i', u)\omega_0(\Phi e_i, u) + \omega_0(v_i', u)\omega_0(\Phi e_{n+1}, u)) \right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i} (\omega_0(\Phi e_i, u)\omega_0(\Phi e_i, u) + \omega_0(\Phi e_{n+1}, u)\omega_0(\Phi e_{n+1}, u)) \right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{a_i} (\omega_0(e_i, \Phi^{-1}u)^2 + \omega_0(e_{n+1}, \Phi^{-1}u)^2) \right),
\end{aligned}$$

and the first statement of the lemma follows.
To prove uniqueness of the n-uple $r_1 \le \ldots \le r_n$ consider the diagonal matrix

$$D(r) = \mathrm{diag}(1/r_1^2, \ldots, 1/r_n^2, 1/r_1^2, \ldots, 1/r_n^2).$$

We must show that if there is a symplectic matrix Φ such that

$$\Phi^{\mathrm{T}} D(r) \Phi = D(r')$$

then $r = r'$. Since $\mathrm{J}_0 \Phi^{\mathrm{T}} = \Phi^{-1} \mathrm{J}_0$ the above identity is equivalent to

$$\Phi^{-1} \mathrm{J}_0 D(r) \Phi = \mathrm{J}_0 D(r').$$

Hence $\mathrm{J}_0 D(r)$ and $\mathrm{J}_0 D(r')$ have the same eigenvalues. But it is easy the check that the eigenvalues of $\mathrm{J}_0 D(r)$ are $\pm\imath/r_1^2, \ldots, \pm\imath/r_n^2$. This proves the lemma. $\square$

Remark 1 *In the case $n = 1$ the existence statement of Lemma 2.4 asserts that every ellipse in $\mathbb{R}^2$ can be mapped into a circle by an area-preserving linear transformation.*

In view of Lemma 2.4 we define the **symplectic spectrum** of an ellipsoid E to be the unique n-uple $r = (r_1, \ldots, r_n)$ with $0 < r_1 \le \ldots \le r_n$ such that E is linearly symplectomorphic to $E(r)$. The spectrum is invariant under linear symplectomorphisms and, in fact, two ellipsoids in $\mathbb{R}^n$, which are centered at 0, are linearly symplectomorphic if and only if they have the same spectrum. Moreover, the volume of an ellipsoid $E \in \mathbb{R}^{2n}$ is given by

$$\mathrm{Vol}(E) = \int_E \frac{\omega_n^n}{n!} = \pi^n \prod_{i=1}^n r_i^2.$$

The following theorem characterizes the linear symplectic width of an ellipsoid in terms of the spectrum.

Theorem 2.5 *Let $E \subset \mathbb{R}^{2n}$ an ellipsoid centered at 0. Then*

$$\mathfrak{W}_L(E) = \sup_{B \subset E} \mathfrak{W}_L(B) = \inf_{E \subset Z} \mathfrak{W}_L(Z),$$

where the supremum runs over all linear symplectic balls contained in E and the infimum runs over all symplectic cylinders containing E. Moreover,

$$\mathfrak{W}_L(E) = \pi r_1^2,$$

where $r = (r_1, \ldots, r_n)$ is the symplectic spectrum associated to E.

Proof. There exists a symplectic matrix $\Phi \in \mathrm{Sp}(2n)$ such that $\Phi E = E(r_1, \ldots, r_n)$. Hence

$$\Phi^{-1} B^{2n}(r_1) \subset E \subset \Phi^{-1} Z^{2n}(r_1)$$

and so

$$\inf_{E \subset Z} \mathfrak{W}_L(Z) \leq \pi r_1^2 \leq \sup_{B \subset E} \mathfrak{W}_L(B).$$

Now suppose that B is a linear symplectic ball of radius r contained in E. Then $\Phi B \subset \Phi E \subset Z^{2n}(r_1)$ and so $r \leq r_1$. Similarly, if Z is a linear symplectic cylinder or radius R containing E then $B^{2n}(r_1) \subset \Phi E \subset \Phi Z$ and so $r_1 \leq R$. Hence

$$\sup_{B \subset E} \mathfrak{W}_L(B) \leq \pi r_1^2 \leq \inf_{E \subset Z} \mathfrak{W}_L(Z).$$

Since $\mathfrak{W}_L(E) = \sup_{B \subset E} \mathfrak{W}_L(B)$ this prove the theorem. $\square$

We finish this section by the following characterization of linear symplectic or anti-symplectic maps.

Theorem 2.6 *Let $\Phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a linear map. Then the following are equivalent.*
(i) Φ preserves the linear width of ellipsoids centered at 0.
(ii) The matrix Φ is either symplectic or anti-symplectic, i.e., $\Phi^* \omega_0 = \pm \omega_0$.

Proof. We have seen that symplectic linear maps preserve the linear symplectic width and it is easy to see that anti-symplectic linear maps do. Now assume (i). We shall prove that Φ has the nonsqueezing property. To see this let B be a linear symplectic ball or radius r and Z be a linear symplectic cylinder of radius R such that

$$\Phi B \subset Z.$$

Then it follows from the monotonicity property of the linear symplectic width that

$$\pi r^2 = \mathfrak{W}_L(B) = \mathfrak{W}_L(\Phi B) \leq \mathfrak{W}_L(Z) = \pi R^2$$

and hence $r \leq R$. It also follows from (i) that Φ must be nonsingular because otherwise the image of the unit ball under Φ would have linear symplectic width zero. Moreover, Φ^{-1} also satisfies (i) because

$$\mathfrak{W}_L(\Phi^{-1} E) = \mathfrak{W}_L(\Phi \Phi^{-1} E) = \mathfrak{W}_L(E)$$

for every ellipsoid E which is centered at zero. Thus we have proved that both Φ and Φ^{-1} have the nonsqueezing property and in view of Theorem 2.4 this implies that Φ is either symplectic or anti-symplectic. $\square$

3 Symplectic manifolds and Hamiltonian flows

A **symplectic structure** on a manifold M is non-degenerate closed 2-form $\omega \in \Omega^2(M)$, i.e., ω is a differential 2-form such that:

1. for any $x \in M$, (T_xM, ω_x) is a symplectic vector space,
2. $d\omega = 0$.

The couple (M, ω) is called **symplectic manifold**.
Let (M, ω) be symplectic manifold. The nondegeneracy implies to the existence of a canonical isomorphism between the tangent and the cotangent bundle, namely,

$$\omega^\flat : TM \longrightarrow T^*M : \quad u \longrightarrow i_u\omega = \omega(u, .).$$

In particular, for any function $H \in C^\infty(M)$, there exists a unique vector field denoted by X_H such that

$$i_{X_H}\omega = dH. \tag{5}$$

The vector field X_H is called **Hamiltonian vector field** associated to H. On the other hand, the nondegeneracy is equivalent to the fact that the maximal form $\Omega = \wedge^n\omega$ is a volume form and hence any symplectic manifold is orientable. A **symplectomorphism** of (M, ω) is a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi^*\omega = \omega$. We denote the group of symplectomorphisms by $\mathrm{Symp}(M, \omega)$. A vector field X is called **symplectic** if its flow preserves ω, i.e., the Lie derivative of ω is the direction of X. Note that according to the Cartan's formula

$$\mathcal{L}_X\omega = di_X\omega + i_Xd\omega$$

and since $d\omega = 0$, X is symplectic if and only if $i_X\omega$ is closed. We denote by $\mathcal{X}(M, \omega)$ the space of symplectic vector fields. It is obvious that any Hamiltonian vector field is symplectic.
The next result shows that, when M is closed (compact without boundary), $\mathcal{X}(M, \omega)$ is the Lie algebra of the group $\mathrm{Symp}(M, \omega)$.

Proposition 3.1 *Let (M, ω) be a closed symplectic manifold. Let (X_t) be a smooth family of vector fields on M and $(\phi_t) \in \mathrm{Diff}(M)$ the smooth family of diffeomorphisms generated by (X_t) via*

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t \quad \textit{and} \quad \phi_0 = \mathrm{id}.$$

Then $\phi_t \in \mathrm{Symp}(M, \omega)$ for every t if and only if $X_t \in \mathcal{X}(M, \omega)$. Moreover, if $X, Y \in \mathcal{X}(M, \omega)$ $[X, Y] \in \mathcal{X}(M, \omega)$ and

$$i_{[X,Y]}\omega = dH \quad \textit{where} \quad H = \omega(X, Y).$$

Proof. The first statement follows from the relation

$$\frac{d}{dt}\phi_t^*\omega = \phi_t^*\left(di_{X_t}\omega + i_{X_t}d\omega\right) = \phi_t^* di_{X_t}\omega.$$

On the other hand, the relations

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X \circ \mathcal{L}_Y\omega - \mathcal{L}_Y \circ \mathcal{L}_X\omega \quad \text{and} \quad i_{[X,Y]}\omega = \mathcal{L}_Y i_X\omega + i_Y\mathcal{L}_X\omega$$

imply in a obvious way the second statement. □

Example 1 *1. The standard model of a symplectic manifold is the Euclidean space $\mathbb{R}^{2n}$ endowed with its canonical symplectic form*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the canonical linear coordinates of $\mathbb{R}^{2n}$.

2. Any oriented surface S endowed with its area form is a symplectic manifold. For instance the 2-sphere S^2 endowed with the 2-form

$$\omega((x,u),(x,v)) = \langle x, u \times v\rangle$$

is a symplectic manifold.

3. **The canonical symplectic structure of the cotangent bundle.** *Let L be a smooth manifold, consider T^*L the total space of its cotangent bundle and denote by $\pi : T^*L \longrightarrow L$ the canonical projection. The* **Liouville form** *in T^*L is the differential 1-form λ in T^*L given by*

$$\lambda(Z_\alpha) = \alpha(T_\alpha\pi(Z_\alpha)),$$

where $\alpha \in T^*L$ and $Z_\alpha \in T_\alpha(T^*L)$. Let $(q_1, \ldots, q_n)$ be a coordinates system on L and $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ the associated coordinates system on T^*L. Then

$$\lambda = \sum_{i=1}^{n} p_i dq_i.$$

This relation implies that

$$d\lambda = \sum_{i=1}^{n} dp_i \wedge dq_i$$

and hence $(T^*L, d\lambda)$ is a symplectic manifold. This symplectic structure on T^*L is called canonical.

Darboux's Theorem asserts that there is no local invariant in symplectic geometry, more precisely, in a given dimension all symplectic forms are locally diffeomorphic.

Theorem 3.1 *Let (M,ω) be a symplectic manifold and $m \in M$. Then there exists a coordinates system $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Such coordinates are called **Darboux's coordinates**.

Proof. According to Theorem 2.1 there is a coordinates system $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ defined on an open set U containing m such that if $\omega_1 = \sum_{i=1}^{n} dq_i \wedge dp_i$ then

$$\omega(m) = \omega_1(m).$$

Moreover, since $\omega_1 - \omega_0$ is closed there exists $\sigma \in \Omega^1(U)$ such that

$$d\sigma = \omega_1 - \omega_0.$$

For $t \geq [0,1]$ put $\omega_t = \omega + td\sigma$. Since $\omega_t(m)$ is nondegenerate and $[0,1]$ is compact, we can choose U such that ω_t is nondegenerate on U for every $t \geq [0,1]$. We consider now the family of vector fields (X_t) defined by

$$i_{X_t}\omega_t = -\sigma$$

and Φ_t the family of diffeomorphisms defined by

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t \quad \text{and} \quad \Phi_0 = \mathrm{id}.$$

Since $X_t(m) = 0$ for every $t \in [0,1]$ we can shrink U if necessary to get Φ_t defined for every $t \in [0,1]$ and $\Phi_t(U) \subset U$. Now

$$\begin{aligned} \frac{d}{dt}\Phi_t^*\omega_t &= \Phi_t^*\left(\frac{d}{dt}\omega_t + i_{X_t}d\omega_t + di_{X_t}\omega_t\right) \\ &= \Phi_t^*\left(d\sigma - d\sigma\right) = 0, \end{aligned}$$

and hence $\Phi_1^*\omega_1 = \omega$ and the theorem follows. □

A **Hamiltonian system** is a triple (M,ω,H) where (M,ω) is a symplectic manifold and H a function on M. The Hamiltonian vector field X_H

associated to H has a flow called **Hamiltonian flow** and its integral curves are solution of

$$\dot{x}(t) = X_H(x(t)).$$

If $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are Darboux's coordinates then this differential system is equivalent to

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \quad \text{and} \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n. \tag{6}$$

Example 2 *The harmonic oscillator is the Hamiltonian system* $(\mathbb{R}^2, \omega_0, H)$ *with*

$$H(x,y) = \frac{1}{2}(x^2 + y^2).$$

The differential system (6) *is written*

$$\dot{x} = y \quad \textit{and} \quad \dot{y} = -x$$

which equivalent to

$$\dot{x} = y \quad \textit{and} \quad \ddot{x} = -x.$$

The corresponding Hamiltonian flow is given by

$$\Phi_t(x,y) = (x\cos t + y \sin t, -x \sin t + y \cos t).$$

4 The Hofer-Zehnder Capacity

In this final section we establish the existence of the Hofer-Zehnder capacity and hence prove the Gromov's nonsqueezing theorem. This capacity is based on properties of the periodic orbits of Hamiltonian flows on a symplectic manifold (M, ω) and was introduced in [5].

Let (M, ω) be a symplectic manifold. Denote the set of all nonnegative Hamiltonian functions which are compactly supported on the interior of M and which attain their maximum on some open set by

$$\mathcal{H}(M) = \left\{ H \in C_0^\infty(\mathrm{int} M) | H \geq 0, H_{|U} = \sup H \text{ form some open set } U \right\}.$$

For every function H consider the time-independent Hamiltonian flow $\phi_H^t \in \mathrm{Symp}^c(M, \omega)$ generated by the Hamiltonian vector field X_H. An orbit $x(t) = \phi_H^t(t)$ is called T-**periodic** if $x(t+T) = x(t)$ for every $t \in \mathbb{R}$. Call a function

$H \in \mathcal{H}(M)$ **admissible** if the corresponding Hamiltonian flow has no nonconstant T-periodic orbit with period $T \leq 1$. In other word, every nonconstant periodic orbit has period > 1. Denote the set of admissible Hamiltonian functions by

$$\mathcal{H}_{\mathrm{ad}}(M, \omega) = \{H \in \mathcal{H}(M) |\ H \text{ admissible}\}.$$

The following lemma shows that for every Hamiltonian function $H \in \mathcal{H}(M)$ the function ϵH is admissible for $\epsilon > 0$ sufficiently small. Roughly speaking, if a vector field is small then its orbits are slow and hence the period is long.

Lemma 4.1 *Let $x(t) = x(t+T) \in \mathbb{R}^m$ be a periodic solution of the differential equation*

$$\dot{x}(t) = f(x),$$

where $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is continuously differentiable. If

$$T. \sup_x \|df(x)\| < 1$$

then $x(t)$ is constant.

Proof. Since $x(0) = x(T)$ an easy calculation shows that

$$\dot{x}(t) = \int_0^t \frac{s}{T} \ddot{x}(s) ds + \int_t^T \frac{s-T}{T} \ddot{x}(s) ds.$$

This implies

$$|\dot{x}(t)| \leq \int_0^T |\ddot{x}(s)| ds \leq \sqrt{T} \|\ddot{x}\|_{L^2[0,T]}$$

and hence

$$\|\dot{x}\|_{L^2} \leq T \|\ddot{x}\|_{L^2}.$$

Note denote $\epsilon = \sup \|df(x)\|$ and note that

$$|\ddot{x}| \leq \|df(x)\|.|\dot{x}| \leq \epsilon |\dot{x}|.$$

Hence

$$\|\ddot{x}\|_{L^2} \leq \epsilon \|\dot{x}\|_{L^2} \leq \epsilon T \|\ddot{x}\|_{L^2}.$$

Since $\epsilon T < 1$ it follows $\ddot{x}(t) \equiv 0$. Hence $\dot{x}(t)$ is constant and periodic and hence $x(t)$ is constant. $\square$

The **Hofer-Zehnder capacity** of (M, ω) is defined by

$$\mathfrak{c}_{HZ}(M, \omega) = \sup_{H \in \mathcal{H}_{\mathrm{ad}}(M, \omega)} \|H\|$$

where $\|H\|$ is the **Hofer norm** given by

$$\|H\| = \sup_{x\in M} H(x) - \inf_{x\in M} H(x).$$

One can deduce easily from Lemma 4.1 that for every nonempty symplectic manifold (M,ω), $\mathfrak{c}_{HZ}(M,\omega) > 0$.

The following theorem is due to Hofer and Zehnder [5].

Theorem 4.1 *The map $(M,\omega) \mapsto \mathfrak{c}_{HZ}(M,\omega)$ satisfies the monotonicity, conformality and normalization axioms of symplectic capacity. Moreover,*

$$\mathfrak{c}_{HZ}(B^{2n}(r),\omega_0) = \mathfrak{c}_{HZ}(Z^{2n}(r),\omega_0) = \pi r^2$$

for every $r > 0$.

The proof of this theorem rests on the following existence result for periodic orbits of Hamiltonian differential equation in $\mathbb{R}^{2n}$ which a proof will be given in the last section.

Theorem 4.2 *Assume $H \in \mathcal{H}(Z^{2n}(1))$ with $\sup H > \pi$. Then the Hamiltonian flow of H has a nonconstant periodic orbit of period 1.*

Proof of Theorem 4.1.

- **Monotonicity.** Let $\phi : (M_1,\omega_1) \longrightarrow (M_2,\omega_2)$ be a symplectic embedding with $\dim M_1 = \dim M_2$. If $H_1 : M_1 \longrightarrow \mathbb{R}$ is a compactly supported function then there is a unique compactly supported function $H_2 : M_2 \longrightarrow \mathbb{R}$ such that H_2 vanishes on $M_2 - \phi(M_1)$ and $H_1 = H_2 \circ \phi$. Since H_1 is compactly supported the function H_2 is smooth. Since ϕ intertwine the Hamiltonian flows of H_1 et H_2 there is a one-to-one correspondence of nonconstant periodic orbits of these flow. Hence

$$\begin{aligned} \mathfrak{c}_{HZ}(M_1,\omega_1) &= \sup_{H_1\in\mathcal{H}_{\mathrm{ad}}(M_1,\omega_1)} \|H_1\| \\ &= \sup_{\substack{H_2\in\mathcal{H}_{\mathrm{ad}}(M_2,\omega_2)\\ \mathrm{supp}(H_2)\subset\phi(M_1)}} \|H_2\| \\ &\leq \mathfrak{c}_{HZ}(M_2,\omega_2). \end{aligned}$$

  This proves monotonicity.

- **Conformality.** Since the Hamiltonian vector field of H with respect to ω agree with the Hamiltonian field of λH with respect to $\lambda\omega$ and hence

$$\mathcal{H}_{\mathrm{ad}}(M,\lambda\omega) = \{\lambda H | H \in \mathcal{H}_{\mathrm{ad}}(M,\omega)\}$$

  and conformality follows.

- **Non triviality.** We shall now prove the inequality $\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) \geq \pi$. Let $\epsilon > 0$ and choose a smooth function $f : [0, 1] \longrightarrow \mathbb{R}$ such that

  $$\forall r,\ -\pi < f'(r) \leq 0,\ f(r) = \pi - \epsilon, \text{ for } r \text{ near } 0 \quad \text{and} \quad f(r) = 0 \text{ for } r \text{ near } 1.$$

  Define $H(z) = f(|z|^2)$ for $z \in B^{2n}(1)$. Then $H \in \mathcal{H}(B^{2n})$ and $\|H\| = \pi - \epsilon$. We must prove now that H is admissible. But the orbits of the Hamiltonian flow are easy to calculate explicitly. According to (6), the Hamiltonian differential equation of H is of the form

  $$\dot{x} = 2f'(|z|^2)y \quad \text{and} \quad \dot{y} = -2f'(|z|^2)x$$

  and it follows that $r = |z(t)|^2$ is constant along the solutions. In complex notation $z = x + \imath y$ the solutions are $z(t) = \exp(-2\imath f'(r)t)z_0$ and are all periodic. They are nonconstant whenever $f'(r) \neq 0$ and in this case th period is $T = \frac{\pi}{f'(r)} > 1$. Hence for every $\epsilon > 0$ there is an admissible Hamiltonian function $H \in \mathcal{H}(B^{2n})$ with $\|H\| = \pi - \epsilon$ and this proves the inequality

  $$\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) \geq \pi.$$

  Now Theorem 4.2 asserts that for every $H \in \mathcal{H}(Z^{2n}(1))$ with $\|H\| > \pi$ the corresponding Hamiltonian flow has nonconstant periodic orbit of period 1. Hence any such function is not admissible and this implies

  $$\mathfrak{c}_{HZ}(Z^{2n}(1), \omega_0) \leq \pi.$$

  By the monotonicity axiom we have

  $$\mathfrak{c}_{HZ}(B^{2n}(1), \omega_0) = \mathfrak{c}_{HZ}(Z^{2n}(1), \omega_0) = \pi$$

  and this proves the theorem. □

We shall now restrict the discussion to subsets of $\mathbb{R}^{2n}$. These subsets are not required to be open, i.e, they are not required to be manifolds. A symplectic embedding $\phi : A \longrightarrow \mathbb{R}^{2n}$ defined on an arbitrary subset $A \subset \mathbb{R}^{2n}$ is by definition a map which extends to a symplectic embedding of an open neighborhood of A. Now a **symplectic capacity** $\mathfrak{c}$ on $\mathbb{R}^{2n}$ assigns a number $\mathfrak{c}(A) \in [0, \infty[$ to every subset $A \subset \mathbb{R}^{2n}$ such that the following holds:

- (**Monotonicity**) If there is a symplectomorphism ϕ of $\mathbb{R}^{2n}$ such that $\phi(A) \subset B$ then $\mathfrak{c}(A) \leq \mathfrak{c}(B)$.

- (**Conformality**) $\mathfrak{c}(\lambda A) = \lambda^2 \mathfrak{c}(A)$.

- **(Non triviality)** $\mathfrak{c}(B^{2n}(1)) > 0$ and $\mathfrak{c}(Z^{2n}(1)) < \infty$.

For every subset $A \subset \mathbb{R}^{2n}$ define

$$\mathfrak{w}_G(A) = \sup\left\{\pi r^2 | B^{2n}(r) \text{ embeds symplectically in} A\right\}$$

and

$$\overline{\mathfrak{w}}_G(A) = \inf\left\{\pi r^2 | A \text{ embeds symplectically in} Z^{2n}(r)\right\}$$

It follows again from Gromov's nonsqueezing theorem that $\mathfrak{w}_G(A)$ and $\overline{\mathfrak{w}}_G$ satisfy the axioms of a symplectic capacity on $\mathbb{R}^{2n}$. If $\mathfrak{c}$ is any other capacity on $\mathbb{R}^{2n}$ we have

$$\mathfrak{w}_G(A) \leq \mathfrak{c}(A) \leq \overline{\mathfrak{w}}_G(A), \tag{7}$$

for every subset $A \in \mathbb{R}^{2n}$.

Example 3 *Recall from Lemma 2.4 that given any ellipsoid*

$$E = \left\{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} x_i x_j \leq 1\right\}$$

there is a linear symplectomorphism $\Phi \in \mathrm{Sp}(2n)$ *such that*

$$\Phi(E) = E(r) := \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} | \sum_{j=1}^{n} \frac{x_j^2 + y_j^2}{r_j^2} \leq 1\right\},$$

for some n-uple $r = (r_1, \ldots, r_n)$ *with* $0 < r_1 \leq \ldots \leq r_n$. *Moreover, r is entirely determined by E. Since*

$$B^{2n}(r_1) \subset \Phi E \subset Z^{2n}(r_1)$$

it follows that

$$\mathfrak{c}(E) = \pi r_1^2 = \mathfrak{w}_L(E)$$

for every symplectic capacity $\mathfrak{c}$ *which satisfies* (1). *Here* $\mathfrak{w}_L$ *denotes the linear symplectic width and the last equation follows from Theorem 2.5.*

A symplectomorphism of a symplectic manifold (M, ω) is a diffeomorphism ϕ such that $\phi^*\omega = \omega$. This definition involves the first derivatives of ϕ and so cannot be generalized in an obvious way to homeomorphisms. This in contrast to the volume-preserving case: a diffeomorphism preserves a volume form if and only if it preserves the corresponding measure. Eliasberg in [3, 4] and, independently, Ekeland-Hofer in [2] realized that one can use capacities in a similar way to give an alternative definition of a symplectomorphism which does not involve derivatives. Their observation is summarized in the following proposition.

Proposition 4.1 *Let $\phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a diffeomorphism and $\mathfrak{c}$ be a symplectic capacity which satisfies* (1)*. Then the following are equivalent.*
(*i*) *ϕ preserves the capacity if ellipsoids, i.e., $\mathfrak{c}(\phi(E)) = \mathfrak{c}(E)$ for every ellipsoids E in $\mathbb{R}^{2n}$.*
(*ii*) *ϕ is either a symplectomorphism or anti-symplectomorphism, i.e., $\phi^*\omega = \pm\omega$.*

Here we consider ellipsoids with arbitrary center. It follows from the definition of a symplectic capacity that every symplectomorphism and every anti-symplectomorphism preserves the symplectic capacity of ellipsoids. For anti-symplectomorphism one needs the additional elementary fact that for every ellipsoid there exists an anti-symplectomorphism which maps this ellipsoid to itself. In Theorem 2.6 we have shown that, conversely, every linear map which preserves the linear symplectic width of ellipsoids is either symplectic or anti-symplectic. Proposition 4.1 is the nonlinear version of this result. The proof is elementary. The only deep observation is the existence of a symplectic capacity. The proof is based on the following lemma.

Lemma 4.2 *Let $\phi_m : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ a sequence of homeomorphisms converging to a homeomorphism $\phi; \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, uniformly on compact sets. Assume that ϕ_m preserves the capacity of ellipsoids for every m. Then ϕ preserves the capacity of ellipsoids.*

Proof. without loss of generality we consider only ellipsoids centered at zero. We first prove that for every ellipsoids E and every positive number $\lambda < 1$ there exists a $m_0 > 0$ such that for every $m \geq m_0$

$$\phi_m(\lambda E) \subset \phi(E) \subset \phi_m(\lambda^{-1}E). \tag{8}$$

To see this, abbreviate $f_m = \phi^{-1} \circ \phi_m$. Then f_m and f_m^{-1} converges to the identity, uniformly on compact sets. So the inclusions $f_m(\lambda E) \subset E$ and $f_m^{-1}(\lambda E) \subset E$ are obvious for large m and (8) follows. This equation now implies that

$$\lambda^2\mathfrak{c}(E) \leq \mathfrak{c}(\phi(E)) \leq \lambda^{-2}\mathfrak{c}(E).$$

Since $\lambda < 0$ was chosen arbitrarily close to 1 it follows that ϕ preserves the capacity of ellipsoids. □

Proof of Proposition 4.1. Assume (i). Then the maps

$$\phi_t(z) = \frac{1}{t}\phi(tz)$$

are diffeomorphisms of $\mathbb{R}^{2n}$ which preserve the capacity of ellipsoids and they converge, uniformly on compact sets the the linear map $\Phi = d\phi(0)$. Hence by Lemma 4.2, Φ preserves the capacity of ellipsoids. We have shown (*cf.* Example 3) that the capacity of ellipsoids agrees with its symplectic width. It follows from Theorem 2.6 that $\Phi^*\omega_0 = \pm\omega_0$. The same holds when Φ is replaced by $d\phi(z)$ for any $z \in \mathbb{R}^{2n}$ and, by continuity, the sign is independent of z. Thus we have proved that (i) implies (ii). The converse is obvious. $\square$

Proposition 4.1 gives rise to the definition of a **symplectic homeomorphism.** Let n be odd. Then an orientation-preserving homeomorphism ϕ of $\mathbb{R}^{2n}$ is said to be symplectic if, for some capacity $\mathfrak{c}$ on subset of $\mathbb{R}^{2n}$, and all sufficiently small ellipsoids E we have $\phi(\mathfrak{c}(E)) = \mathfrak{c}(E)$. If n is even, ϕ is said to be symplectic if the homeomorphism $\phi \times \mathrm{id}$ of $\mathbb{R}^{2n+2}$ satisfies the previous conditions.
One can translate this definition to an arbitrary symplectic manifold using Darboux's theorem. But they are many open questions. For example, if ϕ preserves the capacity of all small ellipsoids, mus it also preserve the capacity of large ellipsoids? Must these symplectic homeomorphisms preserve volume?

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